

# RING EXTENSIONS INVARIANT UNDER GROUP ACTION

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ABSTRACT. Let  $G$  be a subgroup of the automorphism group of a commutative ring with identity  $R$ . Let  $S$  be a subring of  $R$  such that  $S$  is invariant under the action by  $G$ . We show  $S^G \subseteq R^G$  is a minimal ring extension whenever  $S \subseteq R$  is a minimal extension. Of the two types of minimal ring extensions, integral and integrally closed, both of these properties are passed from  $S \subseteq R$  to  $S^G \subseteq R^G$ . An integrally closed minimal ring extension is a flat and epimorphic extension as well as a normal pair. We show each of these properties also pass from  $S \subseteq R$  to  $S^G \subseteq R^G$ .

March 27, 2014

## 1. INTRODUCTION

All rings herein are commutative with identity, and all homomorphisms and subrings are unital. We denote by  $\text{Reg}(R)$  the set of regular elements;  $\text{Spec}(R)$  the set of prime ideals;  $\text{Min}(R)$  the set of minimal prime ideals;  $\text{Max}(R)$  the set of maximal ideals;  $\text{Rad}_R(I)$  the radical in  $R$  of an ideal  $I \subseteq R$ ;  $\text{tq}(R)$  the total quotient ring of  $R$ ;  $\text{qf}(D)$  the quotient field of a domain  $D$ ; and  $\text{Aut}(R)$  the automorphism group of  $R$ .

Throughout,  $G$  is a subgroup of  $\text{Aut}(R)$ . We say  $G$  acts on  $R$  and denote the fixed ring of this action by  $R^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}$ . We say a property of  $R$  is *(G-)invariant* if  $R^G$  also has the property. Our purpose in this paper is to enhance the popular investigation of which ring-theoretic properties are invariant. As the title of this paper suggests, we determine properties of the ring extension  $S \subseteq R$  that are  $G$ -invariant, meaning the property descends to the fixed subring extension  $S^G \subseteq R^G$ . **Our riding assumption in this work is  $S$  is  $G$ -invariant, i.e.,  $\sigma(S) \subseteq S$  for all  $\sigma \in G$ .** Moreover, it follows  $G$  is a subgroup of  $\text{Aut}(S)$ .

We denote the orbit of  $r \in R$  under  $G$  by  $\mathcal{O}_r$ , i.e.  $\mathcal{O}_r = \{\sigma(r) \mid \sigma \in G\}$ , and we define

$$n_r := |\mathcal{O}_r|, \quad \hat{r} := \sum_{r_i \in \mathcal{O}_r} r_i \quad \text{and} \quad \tilde{r} := \prod_{r_i \in \mathcal{O}_r} r_i.$$

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2010 *Mathematics Subject Classification*. Primary 13A50, 13B20 Secondary 13B21, 13A15.

*Key words and phrases*. Fixed ring, ring of invariants, invariant theory, locally finite, minimal ring extension, flat extension, epimorphic extension, normal pair.

We say  $G$  is *locally finite* if  $\mathcal{O}_r$  is finite for all  $r \in R$ . Given an ideal  $I \subset R$  we denote the orbit of  $I$  under  $G$  by  $\mathcal{O}_I = \{\sigma(I) \mid \sigma \in G\}$ . As in [8, Lemma 2.1], by the First Isomorphism Theorem,  $R/I \cong R/\sigma(I)$ . Clearly,  $R/I$  is a field (domain) if and only if  $R/\sigma(I)$  is a field (domain). Hence,  $I$  is a maximal (prime) ideal if and only if  $\sigma(I)$  is a maximal (prime) ideal.

As in [13], we say  $S \subset R$  is a *minimal ring extension* if there is no ring  $T$  such that  $S \subset T \subset R$ . Clearly, this is true if and only if  $R = S[u]$  for all  $u \in R \setminus S$ . Since  $S \subseteq \bar{S} \subseteq R$ , where  $\bar{S}$  is the integral closure of  $S$  in  $R$ , if  $S \subset R$  is minimal, then either  $S$  is integrally closed in  $R$ , or  $R$  is integral over  $S$  (equivalently,  $R$  is module finite over  $S$ ). In the first case we call  $S \subset R$  an *integrally closed minimal ring extension*, and in the second case, we call it an *integral minimal ring extension*. By [13, Théorème 2.2], if  $S \subset R$  is a minimal ring extension, there exists a unique maximal ideal  $M$  of  $S$  such that  $S_P \cong R_P$  for all  $P \in \text{Spec}(S) \setminus \{M\}$ . This maximal ideal is commonly referred to as the *crucial maximal ideal* of the extension. In the integral case,  $(S :_S R)$  is the crucial maximal ideal, while in the integrally closed case,  $(S :_S R)$  is adjacent to the crucial maximal ideal.

In 1970, Ferrand and Olivier contributed to the groundbreaking work of classifying minimal ring extensions by determining the minimal ring extensions of a field [13]. More recently, Ayache [1] extended this work to integrally closed domains. Shortly thereafter, Dobbs and Shapiro generalized these results further to arbitrary domains in [7] and then later to certain rings with zero-divisors in [9]. In their second paper [9], they completely classify the integral minimal ring extensions of an arbitrary ring, as well as the integrally closed minimal ring extensions of a ring with von Neumann regular total quotient ring. In [19] (cf. [6]), Picavet and Picavet-L'Hermitte give another characterization of integral minimal ring extensions. In [3], Cahen et al. characterize integrally closed minimal ring extensions of an arbitrary ring.

In Section 2, under the assumption  $S \subset R$  is an integral minimal ring extension and  $G$  is locally finite acting on  $R$  (such that  $S$  is  $G$ -invariant), we show  $S^G \subset R^G$  is an integral minimal ring extension under mild hypotheses. To do so we use [19, Theorem 3.3], given in Theorem 2.5 for reference. We illustrate when  $S^G = R^G$  under the aforementioned assumptions. In one example, we use the notion of idealization. Given a ring  $R$  and an  $R$ -module  $M$ , the *idealization (of  $M$  over  $R$ )*  $R(+)M = \{(r, m) \mid r \in R, m \in M\}$  is ring with componentwise addition and multiplication given by  $(r, m)(r', m') = (rr', rm' + r'm)$ . By [5, Theorem 2.4],  $R(+)M$  is a minimal ring extension of  $R$  if and only if  $M$  is a simple  $R$ -module.

In Section 3, we turn to the integrally closed case. In Theorem 3.6, we show arbitrary integrally closed minimal ring extensions are invariant under locally finite group action. This invariance is established in [10, Theorem 3.6] under the assumptions that the base ring is a domain in which  $|G| \in \mathbb{N}$  is a unit. The authors use the characterization of the minimal overrings of an integrally closed domain (that is not a field) by Ayache [1, Theorem 2.4] This

result is generalized by Dobbs and Shapiro [9, Theorem 3.7], and then further generalized by Cahen et al. [3, Theorem 3.5]. The latter authors introduce a new classification of integrally closed minimal ring extensions of an arbitrary ring in terms of rank 1 valuation pairs. For an extension  $S \subset R$  and a prime ideal  $P \subset S$ , in [17] (cf. [3]) Manis defines  $(S, P)$  as a *valuation pair (of  $R$ )* if there exists a valuation  $v$  on  $R$  such that  $S = \{r \in R \mid v(r) \geq 0\}$  and  $P = \{r \in R \mid v(r) > 0\}$ . Equivalently,  $(S, P)$  is a valuation pair of  $R$  if  $S = T$  whenever  $T$  is an intermediate ring containing a prime ideal lying over  $P$ . The *rank* of  $(S, P)$  is the rank of the valuation group. A useful necessary and sufficient condition for  $(S, P)$  to have rank 1 given in [3, Lemma 2.12] is that  $P$  is a critical ideal. Cahen et al. define a *critical ideal (for  $S \subset R$ )* as an ideal  $I \subset S$  such that  $I = \text{Rad}_S((S :_S R))$  for all  $r \in R \setminus S$ . (That is,  $\text{Rad}_S((S :_S r))$  is the same ideal for all  $r \in R \setminus S$ .) While such an ideal may not exist for some extensions, if it does, clearly it is unique.

In Section 4, we show certain ring extensions related to minimal ring extensions are also invariant. It is easy to see the integral and integrally closed properties are invariant. Other related extensions are flat epimorphic extensions and normal pairs. Integrally closed minimal ring extensions are flat epimorphic extensions and normal pairs. In Propositions 4.7, 4.9, and 4.11, we show flat epimorphic, epimorphic, and flat extensions, respectively, are invariant under various assumptions. In Proposition 4.13, we give two sufficient conditions for the normal pair property to be invariant. As in [4], for an extension  $S \subset R$ , we say  $(S, R)$  is a *normal pair* if every intermediate ring is integrally closed in  $R$ .

## 2. INTEGRAL MINIMAL RING EXTENSIONS

We begin with a well-known result that is fundamental in this paper and in much of the work by Dobbs and Shapiro [10], [8], [11]. These papers on invariant theory are a strong influence on our work.

**Lemma 2.1.** *If  $G$  is locally finite, then  $R$  is integral over  $R^G$ .*

In the following lemma we establish several technical results needed for the main result of this section. Proposition 2.3 is also necessary and of independent interest.

**Lemma 2.2.** *Assume  $G$  is locally finite. Let  $M := (S :_S R) \in \text{Max}(S)$  and set  $m := M \cap S^G = M \cap R^G$ .*

- (a) *The conductor  $(S^G :_{S^G} R^G)$  is  $m$ .*
- (b) *The orbit of  $M$  in  $S$  and  $R$  is a singleton set, i.e.  $\mathcal{O}_M = \{M\}$ .*
- (c) *If there exist  $N \in \text{Spec}(R)$  containing  $M$ , then  $M = N \cap S$ .*

*Proof.* (a) Let  $x \in m$ . Then  $x \in S^G$ , and  $xr \in S$ , for all  $r \in R$ . If  $r \in R^G$ , then  $xr \in R^G$ , from which it follows that  $xr \in R^G \cap S = S^G$ . Hence,  $x \in (S^G :_{S^G} R^G)$ . Thus,  $m \subseteq (S^G :_{S^G} R^G)$ . By Lemma 2.1,  $S$  is integral over  $S^G$ . Hence,  $m$  is maximal in  $S^G$ . Thus,  $m = (S^G :_{S^G} R^G)$ . (b) Let

$\sigma \in G$ . Then

$$\sigma(M)R = \sigma(M)\sigma(R) = \sigma(MR) \subseteq \sigma(S) = S.$$

Hence,  $\sigma(M) \subseteq M$ . This is sufficient to show  $\sigma(M) = M$  in  $S$ , since  $\sigma(M)$  is maximal in  $S$ , by [8, Lemma 2.1(b)]. Clearly, if  $\sigma(M) = M$  in  $S$ , then  $\sigma(M) = M$  in  $R$ . (c) Clear, since  $M \in \text{Max}(S)$ .  $\square$

**Proposition 2.3.** *Let  $M \in \text{Max}(S)$  and  $m := M \cap S^G$ . Assume  $G$  is locally finite such that  $\text{char}(S^G/m) \nmid n_s$  for all  $s \in S$ . If  $\mathcal{O}_M = \{M\}$ , then the  $G$ -action extends to  $S/M$  via  $\sigma(s + M) = \sigma(s) + M$ , for  $\sigma \in G$ . Moreover,  $S^G/m \cong (S/M)^G$ .*

*Proof.* The given action of  $G$  on  $S/M$  is well-defined: If  $s + M = t + M$ , then  $\sigma(s) - \sigma(t) \in \sigma(M) = M$ . Hence,  $\sigma(s) + M = \sigma(t) + M$ .

As for the moreover, first note  $m \in \text{Max}(S^G)$ , by Lemma 2.1. Define  $\phi : S^G/m \rightarrow (S/M)^G$  by  $s + m \mapsto s + M$ . Clearly,  $\phi$  is a ring homomorphism. If  $\phi(s + m) = 0 + M$ , then  $s \in M$ . It follows  $s \in M \cap S^G = m$ , so  $s + m = 0 + m$ . Hence,  $\phi$  is injective.

Now let  $s + M \in (S/M)^G$ . Then  $s + M = \sigma(s) + M$  for all  $\sigma \in G$ . Summing  $\mathcal{O}_s$  we have  $n_s s + M = \hat{s} + M$ . Since  $S/M$  is a field,  $s + M = (n_s + M)^{-1}(\hat{s} + M)$ . Similarly, since  $n_s + m \in S^G/m$ , we have  $y + m := (n_s + m)^{-1} \in S^G/m$ . It follows  $y + M = (n_s + M)^{-1}$ . Hence,  $\phi(y\hat{s} + m) = y\hat{s} + M = (n_s + M)^{-1}(\hat{s} + M) = s + M$ . Thus,  $\phi$  is surjective. Hence,  $S^G/m \cong (S/M)^G$ .  $\square$

The technique of averaging the orbit of an element used above to produce  $s + M = (n_s + M)^{-1}(\hat{s} + M)$  is introduced in [2]. We generalize this method in the following lemma.

**Lemma 2.4.** *Assume  $T$  is a ring with locally finite  $G$ -action such that (a)  $T$  is a domain and  $\text{char}(T) \nmid n_t$  for all  $t \in T$ , or (b)  $G$  is finite and  $|G|$  is a unit in  $T$ . Let  $r \in T^G$ . If  $r = r_1 u_1 + r_2 u_2 + \cdots + r_k u_k$  for some  $r_i \in T$  and  $u_i \in T^G$ , then there exist  $m, m_i \in \mathbb{N}$  and  $r'_k \in T^G$  such that  $0 \neq mr = m_1 r'_1 u_1 + m_2 r'_2 u_2 + \cdots + m_k r'_k u_k$ .*

*Proof.* For all  $t \in T$ , fix a subset  $\mathcal{N}_t$  of  $G$  such that for each  $a \in \mathcal{O}_t$  there exists a unique  $\sigma \in \mathcal{N}_t$  such that  $a = \sigma(t)$  (and so  $|\mathcal{N}_t| = |\mathcal{O}_t| = n_t$ ).

First we show if

$$(1) \quad 0 \neq r = t_1 u_1 + \cdots + t_i u_i + r_{i+1} u_{i+1} + \cdots + r_k u_k,$$

where  $r, t_i \in R^G$ , then there exists  $m \in \mathbb{N}$ ,  $r'_{i+1} \in R^G$ , and  $s_j \in R$  such that

$$(2) \quad 0 \neq mr = m(t_1 u_1 + \cdots + t_i u_i) + r'_{i+1} u_{i+1} + s_{i+2} u_{i+2} + \cdots + s_k u_k.$$

Applying each  $\sigma \in \mathcal{N}_{r_{i+1}}$  to (1) and summing establishes (2). In particular,

$$m = n_{r_{i+1}}, \quad r'_{i+1} = \widehat{r}_{i+1}, \quad \text{and} \quad s_j = \sum_{\sigma \in \mathcal{N}_{r_{i+1}}} \sigma(r_j) u_j,$$

for  $i + 2 \leq j \leq k$ . Note  $n_{r_{i+1}} r \neq 0$  under assumption (a). Since  $i = 1$  establishes the base case, the assertion of the lemma now follows from

induction. Under assumption (b), the same argument holds replacing  $\mathcal{N}_{r_{i+1}}$  with  $G$  and  $n_{r_{i+1}}$  with  $|G|$ .  $\square$

We have established the machinery needed to prove the main result of this section. We use the characterization provided below for reference.

**Theorem 2.5.** [19, Theorem 3.3] (cf. [6, Corollary II.2]) *Let  $S \rightarrow R$  be an injective ring homomorphism, with conductor  $(S :_S R)$ . Then  $S \rightarrow R$  is minimal and finite if and only if  $(S :_S R) \in \text{Max}(S)$  and one of the following three conditions holds:*

- (a) **Inert case:**  $(S :_S R) \in \text{Max}(R)$  and  $S/(S :_S R) \rightarrow R/(S :_S R)$  is a minimal field extension.
- (b) **Decomposed case:** There exist  $N_1, N_2 \in \text{Max}(R)$  such that  $(S : R) = N_1 \cap N_2$  and the natural maps  $S/(S :_S R) \rightarrow R/N_1$  and  $S/(S :_S R) \rightarrow R/N_2$  are each isomorphisms.
- (c) **Ramified case:** There exists  $N \in \text{Max}(R)$  such that  $N^2 \subseteq (S :_S R) \subset N$ ,  $[R/(S :_S R) : S/(S :_S R)] = 2$  and the natural map  $S/(S :_S R) \rightarrow R/N$  is an isomorphism.

**Theorem 2.6.** *Let  $S \subset R$  be an integral minimal extension with crucial maximal ideal  $M = (S :_S R)$ . Assume  $G$  locally finite such that  $S^G \neq R^G$  and  $\text{char}(S^G/(M \cap S^G)) \nmid n_s$ , for all  $s \in S$ . Then  $S^G \subset R^G$  is a minimal extension of the same type as  $S \subset R$ . Moreover, the crucial maximal ideal of  $S^G \subset R^G$  is  $(S^G :_{S^G} R^G)$ .*

*Proof.* Throughout the argument, set  $m := (S^G :_{S^G} R^G)$ , whence  $m = M \cap S^G = m \cap R^G$ , by Lemma 2.2(a).

**Inert case:** By Theorem 2.5(a),  $M \in \text{Max}(R)$  and  $S/M \rightarrow R/M$  is a minimal field extension. By Lemma 2.2(b) and Proposition 2.3, we may pass to  $S/M \subset R/M$ . Replacing  $S/M \subset R/M$  with  $S \subset R$ , we show  $S^G \subset R^G$  is a minimal field extension. Clearly this is true if and only if for all  $u \in R^G \setminus S^G$ ,  $R^G = S^G[u]$ . If  $u \in R^G \setminus S^G$ , then  $u \in R \setminus S$ , so  $R = S[u]$ . Let  $r \in R^G$ . Then  $r = s_k u^k + \cdots + s_1 u + s_0$ , for some  $k \in \mathbb{N}$  and  $s_i \in S$ . By Lemma 2.4, there exist  $m, m_i \in \mathbb{N}$  and  $s'_i \in S^G$  such that  $0 \neq mr = m_k s'_k u^k + \cdots + m_1 s'_1 u + m_0 s'_0$ . Since  $S^G$  is a field, we have  $r = m^{-1}(m_k s'_k u^k + \cdots + m_1 s'_1 u + m_0 s'_0) \in S^G[u]$ . Hence,  $S^G \subset R^G$  is a minimal field extension. By Theorem 2.5(a), our original (before passing to the quotient ring extension)  $S^G \subset R^G$  is an inert integral minimal extension with crucial maximal ideal  $m = (S^G :_{S^G} R^G)$ .

**Decomposed case:** By Theorem 2.5(b), there exist  $N_1, N_2 \in \text{Max}(R)$  such that  $M = N_1 \cap N_2$  and the natural maps  $S/M \rightarrow R/N_1$  and  $S/M \rightarrow R/N_2$  are isomorphisms. Set  $n_1 := N_1 \cap R^G$  and  $n_2 := N_2 \cap R^G$ . Since  $R$  is integral over  $R^G$ ,  $n_1, n_2 \in \text{Max}(R^G)$ . Clearly,  $m = M \cap R^G = (N_1 \cap N_2) \cap R^G = n_1 \cap n_2$ .

Define  $\phi : S^G/m \rightarrow R^G/n_1$  as the natural map  $s + m \mapsto s + n_1$ . Suppose  $\phi(s + m) = 0 + n_1$  for some  $s \in S^G$ . Then  $s \in n_1 \cap S^G$ , but, by Lemma 2.2(c),  $n_1 \cap S^G = m$ . Hence,  $s + m = 0 + m$ . Thus,  $\phi$  is injective.

To show  $\phi$  is surjective, we first note the  $G$ -action extends to  $R/N_1$ , since it extends to  $S/M$  and  $S/M \cong R/N_1$ . From Lemma 2.2(b) and Proposition 2.3, we have  $S^G/m \cong (S/M)^G \cong (R/N_1)^G$ . Let  $0 + n_1 \neq r + n_1 \in R^G/n_1$ . Then  $0 + N_1 \neq r + N_1 \in (R/N_1)^G$ . (Clearly it is fixed, and if  $r \in N_1$ , then  $r \in N_1 \cap R^G = n_1$  - contradiction.) Since  $S^G/m \cong (R/N_1)^G$  (via composition of the natural maps), there exists  $s + m \in S^G/m$  such that  $s + m \mapsto s + M \mapsto s + N_1 = r + N_1$ . It follows  $(s - r) \in N_1 \cap R^G = n_1$ . Hence,  $\phi(s + m) = s + n_1 = r + n_1$ . Thus,  $\phi$  is surjective, so  $S^G/m \cong R^G/n_1$ . The same argument applies to show  $S^G/m \cong R^G/n_2$ . By Theorem 2.5(b),  $S^G \subset R^G$  is a decomposed integral minimal extension with crucial maximal ideal  $m = (S^G :_{S^G} R^G)$ .

**Ramified case:** By Theorem 2.5(c), there exists  $N \in \text{Max}(R)$  such that  $N^2 \subseteq M \subset N$ ,  $[R/M : S/M] = 2$  and the natural map  $S/M \rightarrow R/N$  is an isomorphism. Set  $n := N \cap R^G$ , and recall  $m = M \cap R^G$ . Clearly,  $n \in \text{Max}(R^G)$  and  $m \subsetneq n$ , since  $m \notin \text{Max}(R^G)$  (since  $M \notin \text{Max}(R)$ ,  $N \in \text{Max}(R)$ , and  $R$  is integral over  $R^G$ ). For the other containment, let  $x \in n^2$ . Then  $x \in N^2$ , so  $x \in M$ . Hence,  $x \in M \cap R^G = m$ . Thus,  $n^2 \subseteq m$ .

Define  $\phi : S^G/m \rightarrow R^G/n$  to be the natural map  $s + m \mapsto s + n$ . Suppose  $\phi(s + m) = 0 + n$  for some  $s \in S^G$ . Then  $s \in n$ , so  $s^2 \in n^2$ . Since  $n^2 \subseteq m$  and  $m$  is prime (maximal) in  $S^G$ , we have  $s \in m$ . (Alternatively,  $s \in n \cap S^G = m$ , by Lemma 2.2(c).) Hence,  $s + m = 0 + m$ . Thus,  $\phi$  is injective.

Next we show  $\phi$  is surjective. Let  $r + n \in R^G/n$ . Then  $r + N \in (R/N)^G$ . Note that, as in the decomposed case, since  $S/M \cong R/N$  via  $s + M \mapsto r + N$ , the  $G$ -action extends to  $R/N$ . From this, Lemma 2.2(b), and Proposition 2.3, it follows  $S^G/m \cong (S/M)^G \cong (R/N)^G$  via  $s + m \mapsto s + M \mapsto s + N$ . Hence, there exists  $s + m \in S^G/m$  such that  $s + m \mapsto s + M \mapsto s + N = r + N$ , from which it follows  $(s - r) \in N \cap R^G = n$ . Hence,  $\phi(s + m) = r + n$ . Thus,  $\phi$  is surjective.

It remains to show  $[R^G/m : S^G/m] = 2$ . Suppose  $[R^G/m : S^G/m] > 2$ , and let  $\{e_1 + m, e_2 + m, e_3 + m\}$  be an  $S^G/m$ -linearly independent set in  $R^G/m$ . Then each  $e_i \notin M$ ; otherwise,  $e_i \in M \cap R^G = m$ . Hence, each  $e_i + M$  is nonzero in  $R/M$ . Since  $[R/M : S/M] = 2$ , without loss of generality we may assume there exist  $s_1 + M, s_2 + M \in S/M$  such that

$$e_3 + M = (s_1 + M)(e_1 + M) + (s_2 + M)(e_2 + M) = s_1 e_1 + s_2 e_2 + M.$$

Applying elements of  $G$  that produce  $\mathcal{O}_{s_1}$  and adding them we have

$$n_{s_1} e_3 + M = \widehat{s}_1 e_1 + \left( \sum_{j=1}^{n_{s_1}} \sigma_j(s_2) \right) e_2 + M,$$

where  $\{\sigma_j(s_1)\}_{j=1}^{n_{s_1}} = \mathcal{O}_{s_1}$ . Defining  $s_3$  to be the above coefficient of  $e_2$  and repeating the above technique with respect to  $s_3$  we have

$$n_{s_3} n_{s_1} e_3 + M = n_{s_3} \widehat{s}_1 e_1 + \widehat{s}_3 e_2 + M.$$

It follows  $n_{s_3}n_{s_1}e_3 - (n_{s_3}\widehat{s}_1e_1 + \widehat{s}_3e_2) \in M \cap R^G = m$ , so

$$n_{s_3}n_{s_1}e_3 + m = n_{s_3}\widehat{s}_1e_1 + \widehat{s}_3e_2 + m.$$

Equivalently,

$$(n_{s_3}n_{s_1} + m)(e_3 + m) = (n_{s_3}\widehat{s}_1 + m)(e_1 + m) + (\widehat{s}_3 + m)(e_2 + m)$$

is an  $S^G/m$ -linear combination of  $e_1 + m, e_2 + m, e_3 + m$  in  $R^G/m$  - contradiction. Hence, there cannot exist in  $R^G/m$  any more than two  $S^G/m$ -linearly independent elements. Thus,  $[R^G/m : S^G/m] \leq 2$ . Note  $R^G/m$  is not a domain, since  $n^2 \subseteq m \subset n$  implies  $m = n$ , if  $m$  is prime. Hence,  $R^G/m \neq S^G/m$ . Thus,  $[R^G/m, S^G/m] = 2$ . By Theorem 2.5(c),  $S^G \subset R^G$  is a ramified integral minimal extension with crucial maximal ideal  $m = (S^G :_{S^G} R^G)$ .  $\square$

**Remark 2.7.** It is necessary to assume  $S^G \neq R^G$  in Theorem 2.6, as illustrated by the following examples.

**Example 2.8.** The fixed rings are equal, even under finite group action, in the following cases:

**Inert case:** Set  $S := \mathbb{R}$ ,  $R := \mathbb{C}$ , and  $G = \{1, \sigma\}$ , where  $\sigma$  is the conjugacy map. Then  $S^G = S = R^G$ .

**Decomposed case:** Assume  $S$  is a field, and set  $R := S \times S$ . By [13, Lemme 1.2(b)],  $S \subset R$  is a minimal extension. Define  $G := \{1, \sigma\}$ , where  $\sigma((s, s)) = (s, -s)$ . Then  $S^G = S = R^G$ .

**Ramified case:** Assume  $S$  is a field, and set  $R := S(+)S$ . By [13, Lemme 1.2(c)],  $S \subset R$  is a minimal extension. Define  $G$  as above. Then  $S^G = S = R^G$ .

### 3. INTEGRALLY CLOSED MINIMAL EXTENSION

In this section, we show that the integrally closed minimal property of the extension  $S \subset R$  is invariant under locally finite  $G$ -action such that  $S^G \neq R^G$ . This generalizes Dobbs' and Shapiro's result [10, Theorem 3.6] that the property is invariant if  $S$  is a domain and if  $|G|$  is finite and a unit in  $S$ . They use Ayache's characterization [1, Theorem 2.4] of minimal extensions (overrings) of an integrally closed domain. Ayache's result has since been generalized by Dobbs and Shapiro [9, Theorem 3.7] and recently further generalized by Cahen et al. [3, Theorem 3.5]. In the latter, the authors give several necessary and sufficient conditions for an arbitrary ring extension to be integrally closed and minimal, which we use to establish Theorem 3.6.

Whereas crucial maximal ideals are historically essential to the study of minimal extensions, Cahen et al. [3] introduce critical ideals and use them extensively in characterizing integrally closed minimal extensions of an arbitrary ring. As previously mentioned, they define a critical ideal for  $S \subset R$  as an ideal  $J \subset S$  such that  $J = \text{Rad}_S((S :_S r))$  for all  $r \in R \setminus S$ . (That is,  $\text{Rad}_S((S :_S r))$  is the same ideal for all  $r \in R \setminus S$ .) They show in

[3, Lemma 2.11] that if an extension has a critical ideal, then the ideal is prime. Moreover, they show that if  $S \subset R$  is a minimal extension, then the critical ideal exists [3, Proposition 2.14(2)] and is maximal [3, Theorem 3.5]. If  $S \subset R$  has a critical ideal, we show  $S^G \subset R^G$  has a critical ideal under any  $G$ -action such that  $S^G \neq R^G$ .

**Lemma 3.1.** *Let  $P$  be the critical ideal of  $S \subset R$ . If  $S^G \neq R^G$ , then  $p := P \cap S^G$  is the critical ideal of  $S^G \subset R^G$ .*

*Proof.* Let  $r \in R^G \setminus S^G$ . Then  $r \in R \setminus S$ . Hence,  $P = \text{Rad}_S((S :_S r))$ , from which it follows

$$p = \text{Rad}_S((S :_S r)) \cap S^G = \text{Rad}_{S^G}((S :_S r) \cap S^G) = \text{Rad}_{S^G}((S^G :_{S^G} r)).$$

Thus,  $p$  is the critical ideal of  $S^G \subset R^G$ .  $\square$

We next show if a critical ideal is maximal, then its orbit (under  $G$ ) is a singleton set.

**Lemma 3.2.** *Suppose  $M = \text{Rad}_S((S :_S r))$ , for all  $r \in R \setminus S$ . If  $M$  is maximal, then  $\sigma(M) = M$  for all  $\sigma \in G$ , i.e.  $\mathcal{O}_M = \{M\}$ .*

*Proof.* Let  $\sigma \in G$  and  $r \in R \setminus S$ . Note  $\sigma^{-1}(r) \in R \setminus S$ ; otherwise, if  $\sigma^{-1}(r) \in S$ , then  $r = \sigma(\sigma^{-1}(r)) \in \sigma(S) = S$  - contradiction. Hence,  $M = \text{Rad}_S((S :_S \sigma^{-1}(r)))$ . Let  $x \in M$ , and set  $y := \sigma^{-1}(x)$ . Then there exists  $n \in \mathbb{N}$  such that  $x^n r \in S$ , from which it follows  $(\sigma^{-1}(x))^n \sigma^{-1}(r) \in \sigma^{-1}(S) = S$ . Hence,  $y = \sigma^{-1}(x) \in \text{Rad}_S((S :_S \sigma^{-1}(r))) = M$ . Hence,  $x = \sigma(y) \in \sigma(M)$ . Thus,  $M \subseteq \sigma(M)$ . Since  $M$  is maximal,  $M = \sigma(M)$ , as desired.  $\square$

**Remark 3.3.** It is not necessary to assume  $M$  is maximal in the preceding lemma. A similar set-theoretic argument establishes the converse  $\sigma(M) \subseteq M$ .

For an extension  $S \subset R$ , related to critical ideals are valuation pairs. As in the introduction and [17], for  $P \in \text{Spec}(S)$ ,  $(S, P)$  is a valuation pair of  $R$  if there is a valuation  $v$  on  $R$  with  $S = \{r \in R \mid v(r) \geq 0\}$  and  $P = \{r \in R \mid v(r) > 0\}$ . Equivalently,  $(S, P)$  is a valuation pair of  $R$  if  $S = T$  whenever  $T$  is an intermediate ring containing a prime ideal lying over  $P$ . Rank 1 valuation pairs are one of several equivalences of integrally closed minimal extensions given by Cahen et al [3]. As previously mentioned, the rank of a valuation pair  $(S, P)$  of  $R$  is the rank of the valuation group. The following lemma describes the relationship between critical ideals and valuation pairs.

**Lemma 3.4.** [3, Lemma 2.12] *Let  $(S, P)$  be a valuation pair of  $R$ . Then  $S \subset R$  has a critical ideal if and only if  $(S, P)$  has rank 1. Moreover, under these conditions,  $P$  is the critical ideal of  $S \subset R$ .*

Our next result is fundamental to the invariance of integrally closed minimal extensions established in Theorem 3.6.



**Proposition 3.5.** *Assume  $G$  is locally finite such that  $S^G \neq R^G$ . Let  $M \in \text{Max}(S)$  and set  $m := M \cap S^G$ . If  $\mathcal{O}_M = \{M\}$ , then  $(S^G, m)$  is a valuation pair of  $R^G$  whenever  $(S, M)$  is a valuation pair of  $R$ .*

*Proof.* Let  $A$  be a ring such that  $S^G \subseteq A \subseteq R^G$ . Then  $S \subseteq AS \subseteq R$ . First note  $AS$  is integral over  $A$ , since  $S$  is integral over  $S^G$ , hence over  $A$ . Let  $q \in \text{Spec}(A)$  such that  $q \cap S^G = m$ , and let  $Q \in \text{Spec}(AS)$  lie over  $q$ . From

$$m = q \cap S^G = (Q \cap A) \cap S^G = Q \cap S^G = (Q \cap S) \cap S^G$$

it follows  $Q \cap S$  is maximal in  $S$ , by integrality. We claim  $Q \cap S = M$ . Suppose not. Then there exists  $x \in (Q \cap S) \setminus M$ , since  $Q \cap S$  and  $M$  are incomparable (as maximal ideals). It follows  $\tilde{x} \in Q \cap S^G = m = M \cap S^G$ . Hence,  $\sigma(x) \in M$  for some  $\sigma \in G$ . Since  $\mathcal{O}_M = \{M\}$ , we have  $x \in \sigma^{-1}(M) = M$  - contradiction. Hence,  $Q \cap S = M$ . Since  $(S, M)$  is a valuation pair of  $R$ , we have  $AS = S$ . Hence,  $A = S^G$ . Thus,  $(S^G, m)$  is a valuation pair of  $R^G$ .  $\square$

Of the several integrally closed minimal extension equivalences in [3, Theorem 3.5], for  $S \subset R$  we use the condition that there exists a maximal ideal  $M$  such that  $(S, M)$  is a valuation pair for  $R$ . With this equivalence, it follows easily from the preceding results that integrally closed minimal extensions are invariant under locally finite group action.

**Theorem 3.6.** *Assume  $G$  is locally finite. If  $S \subset R$  is a integrally closed minimal extension, then  $S^G \subset R^G$  is an integrally closed minimal extension.*

*Proof.* First we show  $S^G \neq R^G$ . Let  $r \in R \setminus S$ . Then  $\tilde{r} \in R^G$ . If  $\tilde{r} \in S^G$ , then  $\tilde{r} \in S$ . By [13, Proposition 3.1],  $\sigma(r) \in S$  for some  $\sigma \in G$ , whence  $r = \sigma^{-1}(\sigma(r)) \in \sigma^{-1}(S) = S$  - contradiction. Hence,  $\tilde{r} \in R^G \setminus S^G$ . Thus,  $S^G \subsetneq R^G$ .

Let  $M$  be the critical ideal for  $S \subset R$ . By Lemma 3.1,  $m := M \cap S^G$  is the critical ideal for  $S^G \subset R^G$ . Since  $S \subset R$  is a minimal extension, the critical ideal  $M$  is maximal. By Lemma 3.2,  $\mathcal{O}_M = \{M\}$ . By Lemma 3.5,  $(S^G, m)$  is a valuation pair of  $R^G$ . Since  $m$  is the critical ideal of  $S^G \subset R^G$ , this valuation pair has rank 1, by Lemma 3.4. Hence,  $S^G \subset R^G$  is an integrally closed minimal extension, by [3, Proposition 3.5].  $\square$

#### 4. RELATED RESULTS

In this section, we generalize the results of Sections 2 and 3. It is easy to see in Propositions 4.1 and 4.2 that arbitrary integral and integrally closed extensions are invariant. In Propositions 4.7, 4.3, 4.11, 4.9, and 4.13 we exchange stronger assumptions for more general results. In particular, we assume  $G$  is finite and  $|G|$  is a unit.

**Proposition 4.1.** *If  $S \subset R$  is an integral extension and  $G$  is locally finite, then  $S^G \subseteq R^G$  is an integral extension.*

*Proof.* Clear from Lemma 2.1 and by transitivity [14, Theorem 40].  $\square$

**Proposition 4.2.** *If  $S$  is integrally closed in  $R$ , then  $S^G$  is integrally closed in  $R^G$ .*

*Proof.* Let  $u \in R^G$  be integral over  $S^G$ . Then  $u \in R$  is integral over  $S$ . Hence,  $u \in R^G \cap S = S^G$ .  $\square$

As in Theorems 2.6 and 3.6, minimal extensions are invariant under locally finite  $G$ -action. In the former however, we require a certain restriction of characteristic. Assuming  $|G|$  is finite and a unit in the base ring, we can remove this restriction. Of course, if  $G$  is finite, then it is locally finite. Hence, the following result and corollary reassert Theorem 3.6.

**Proposition 4.3.** *Let  $S \subset R$  be a minimal extension. Assume  $G$  is finite such that  $|G|$  is a unit in  $S$  and  $S^G \neq R^G$ . Then  $S^G \subset R^G$  is a minimal extension.*

*Proof.* Let  $u \in R^G \setminus S^G$ . Clearly,  $u \in R \setminus S$ . Hence,  $R = S[u]$ . Let  $r \in R^G$ . Then  $r = s_n u^n + \cdots + s_1 u + s_0$  for some  $s_i \in S$ . Applying the averaging technique introduced in Section 2 we have  $|G|r = \widehat{s}_n u^n + \cdots + \widehat{s}_1 u + \widehat{s}_0$ . Hence,  $r = |G|^{-1}(\widehat{s}_n u^n + \cdots + \widehat{s}_1 u + \widehat{s}_0)$ . Thus,  $R^G = S^G[u]$ , i.e.  $S^G \subset R^G$  is a minimal extension.  $\square$

**Corollary 4.4.** *Under the hypotheses of Proposition 4.3, if  $S \subset R$  is an integral or integrally closed minimal extension, then  $S^G \subset R^G$  is an integral or integrally closed minimal extension, respectively.*

Integrally closed minimal extensions are flat epimorphic extensions, by [13, Théorème 2.2]. Equivalently, flat epimorphisms are *perfect localizations*, so-called because of the following correspondence.

**Theorem 4.5.** [20, Theorem 2.1] *Let  $\phi : A \rightarrow B$  be a ring homomorphism. Then  $\phi$  is a flat epimorphism if and only if the collection  $\mathcal{F} = \{I \subset A \mid \phi(I)B = B\}$  where  $I$  is an ideal in  $A$  is a Gabriel filter, and  $\psi : B \rightarrow A_{\mathcal{F}}$  is an isomorphism such that  $\psi\phi : A \rightarrow A_{\mathcal{F}}$  is the canonical homomorphism. Such a filter is called perfect.*

By [20, Exercise 8, p. 242],  $R$  is a perfect localization of  $S$  if and only if for all  $r \in R$ ,  $(S : r)R = R$ . With this definition and Lemma 4.6 we show perfect localizations (equivalently, flat epimorphic extensions) are invariant in Proposition 4.7.

**Lemma 4.6.** *Assume  $G$  is locally finite. Define  $\mathcal{F} := \{I \subset S \mid IR = R\}$  and  $\mathcal{F}' := \{J \subset S^G \mid JR^G = R^G\}$ . If  $I \in \mathcal{F}$ , then  $I \cap S^G \in \mathcal{F}'$ .*

*Proof.* Note  $I \in \mathcal{F}$  if and only if every  $P \in \text{Spec}(S)$  containing  $I$  is not lain over in  $R$ . Also note  $\mathcal{F}' = \{J \subset S^G \mid JS \in \mathcal{F}\}$ . Let  $I \in \mathcal{F}$  and let  $P \in \text{Spec}(S)$  contain  $(I \cap S^G)S$ . We claim  $I \subseteq \sigma(P)$  for some  $\sigma \in G$ , whence  $PR = \sigma^{-1}(\sigma(P)R) = \sigma^{-1}(\sigma(PR)) = R$ . Let  $x \in I$ . Then  $\tilde{x} \in I \cap S^G$ , so  $\tilde{x} \in P$ . It follows  $\sigma(x) \in P$  for some  $\sigma \in G$ ; equivalently,  $x \in \sigma^{-1}(P)$ . Hence, the claim is satisfied by  $\sigma^{-1}$ , so  $PR = R$ . Thus, every prime containing

$(I \cap S^G)S$  is not lain over in  $R$ . By the first note,  $(I \cap S^G)S \in \mathcal{F}$ , and, by the second note,  $I \cap S^G \in \mathcal{F}'$ , as desired.  $\square$

We are now ready to show perfect localizations (flat epimorphic extensions) are invariant using Lemma 4.6.

**Proposition 4.7.** *Let  $G$  be locally finite, and let  $\mathcal{F}$  and  $\mathcal{F}'$  be as in Lemma 4.6. Then (a) if  $\mathcal{F}$  is a Gabriel filter, then  $\mathcal{F}'$  is a Gabriel filter, and (b) if  $R = S_{\mathcal{F}}$ , then  $R^G = (S^G)_{\mathcal{F}'}$ .*

*Proof.* (a) Suppose  $\mathcal{F}$  is a Gabriel filter. Let  $I \in \mathcal{F}'$ , and let  $J$  be an ideal of  $S^G$ . Then  $IS \in \mathcal{F}$  and  $IS \subseteq JS$ , so  $JS \in \mathcal{F}$ . It follows  $JR = R$ , so  $JR^G = R^G$ , since  $R$  is integral over  $R^G$ . Hence,  $J \in \mathcal{F}'$ . Now let  $I, J \in \mathcal{F}'$ . Then  $IR = R$  and  $JR = R$ . Suppose  $I \cap J \notin \mathcal{F}'$ , i.e.  $(I \cap J)R^G \neq R^G$ . Again by integrality,  $(I \cap J)R \neq R$ . Let  $P \in \text{Spec}(R)$  contain  $(I \cap J)R$ . Then  $I \cap J \subseteq P \cap S^G =: p$ . It follows  $I \subseteq p$  or  $J \subseteq p$ , but then  $IR \subseteq P$  or  $JR \subseteq P$  - contradiction. Hence,  $I \cap J \in \mathcal{F}'$ . Now let  $J$  be an ideal of  $S^G$ , and suppose there exists  $I \in \mathcal{F}'$  such that  $(J :_{S^G} a) \in \mathcal{F}'$  for all  $a \in I$ . We claim  $(JS :_S a) \in \mathcal{F}$  for all  $a \in IS$ , whence  $JS \in \mathcal{F}$ , i.e.  $J \in \mathcal{F}'$ . Let  $a := a_1 s_1 + \cdots + a_n s_n \in IS$ , where  $a_i \in I$  and  $s_i \in S$ . For each  $a_i$ , clearly  $(J :_{S^G} a_i)S \subseteq (JS :_S a_i)$ , so  $(JS :_S a_i) \in \mathcal{F}$ , since  $(J :_{S^G} a_i) \in \mathcal{F}'$ . From  $(JS :_S a_i) \subseteq (JS :_S a_i s_i)$  it follows  $(JS :_S a_i s_i) \in \mathcal{F}$ . Since  $\bigcap_{i=1}^n (JS :_S a_i s_i) \in \mathcal{F}$  and  $\bigcap_{i=1}^n (JS :_S a_i s_i) \subseteq (JS :_S a)$ , we have  $(JS :_S a) \in \mathcal{F}$ , proving the claim. Hence,  $JS \in \mathcal{F}$ , i.e.  $J \in \mathcal{F}'$ , as above. Thus,  $\mathcal{F}'$  is a Gabriel filter.

(b) Now we show  $R^G = (S^G)_{\mathcal{F}'}$  by showing  $R^G$  is a perfect localization of  $S^G$ . Let  $x \in R^G$ . Then  $(S :_S x)R = R$ , since  $R$  is a perfect localization of  $S$ . It follows  $(S :_S x) \in \mathcal{F}$ , and  $(S :_S x) \cap S^G \in \mathcal{F}'$ , by Lemma 4.6. We claim  $(S :_S x) \cap S^G \subseteq (S^G :_{S^G} x)$ , whence  $(S^G :_{S^G} x) \in \mathcal{F}'$ , since  $\mathcal{F}'$  is a Gabriel filter. Let  $y \in (S :_S x) \cap S^G$ . Then  $xy \in S$ , but  $x \in R^G$  and  $y \in S^G$ , so  $xy \in S^G$ . Hence,  $(S :_S x) \cap S^G \subseteq (S^G :_{S^G} x)$ , so  $(S^G :_{S^G} x) \in \mathcal{F}'$  as claimed. (In fact, as the reverse containment clearly holds,  $(S :_S x) \cap S^G = (S^G :_{S^G} x)$ .) Thus,  $(S^G :_{S^G} x)R^G = R^G$ , i.e.  $R^G$  is a perfect localization of  $S^G$ .  $\square$

Since flat epimorphic extensions are invariant, naturally we are interested in the cases of flat extensions and epimorphic extensions. We answer both questions in the positive, under a slightly stronger assumptions. We require two more technical lemmas; the second, Lemma 4.10, we wait to introduce until needed.

**Lemma 4.8.** *Assume  $G$  acts on  $R \otimes_S R$  via  $\sigma(r_1 \otimes_S r_2) = \sigma(r_1) \otimes_S \sigma(r_2)$ . Let  $F(R \times R)$  and  $F(R^G \times R^G)$  be the free abelian groups on  $R \times R$  and  $R^G \times R^G$ , respectively. Let  $H$  and  $H'$  be their respective subgroups satisfying  $(R \otimes_S R)^G = F(R \times R)/H$  and  $R^G \otimes_{S^G} R^G = F(R^G \times R^G)/H'$ . Then  $H' = H \cap F(R^G \times R^G)$ , if (a)  $R$  is a domain, or (b)  $G$  is finite and  $|G|$  is a unit in  $S$ .*

*Proof.* (a) The containment  $H' \subseteq H \cap F(R^G \times R^G)$  is clear. For the reverse containment, let  $n_1 T_1 + \cdots + n_r T_r \in H \cap F(R^G \times R^G)$  for  $T_i \in H$ . We may assume each  $T_i$  is a generator of  $H$ . That is,  $T_i$  is of the form  $(r_1 + r_2, r) - (r_1, r) - (r_2, r)$ ,  $(r, r_1 + r_2) - (r, r_1) - (r, r_2)$ , or  $(sr_1, r_2) - (r_1, sr_2)$ , where  $r, r_1, r_2 \in R$  and  $s \in S$ . Since  $F(R^G \times R^G)$  is free, each  $T_i \in F(R^G \times R^G)$ , so in fact  $r, r_1, r_2, sr_1, sr_2 \in R^G$ . Hence, if  $T_i$  is of the first or second aforementioned forms, then  $T_i \in H'$ . If  $T_i = (sr_1, r_2) - (r_1, sr_2)$ , then  $sr_1 = \sigma(sr_1) = \sigma(s)r_1$  for any  $\sigma \in G$ . Since  $R$  is a domain,  $\sigma(s) = s$ , i.e.  $s \in S^G$ . Hence,  $T_i \in H'$ , which proves the reverse containment.

(b) In the above, we only require  $R$  to be a domain in order to show that if  $sr_1 \in R^G$ , then  $s \in S^G$ . If  $|G|$  is finite and is a unit in  $S$ , we may apply the averaging technique introduced in Section 2. Since  $sr_1 \in R^G$ , this yields  $sr_1 = |G|^{-1} \hat{s} r_1$ . Similarly,  $sr_2 = |G|^{-1} \hat{s} r_2$ . Substituting  $|G|^{-1} \hat{s} r_1$  for  $sr_1$  and  $|G|^{-1} \hat{s} r_2$  for  $sr_2$  in (a) shows  $H' = H \cap F(R^G \times R^G)$ . (Note, we do not need  $s \in S^G$  but only  $sr_i = s' r'_i$  for some  $s' \in S^G$  and  $r'_i \in R^G$  for each  $i$ .)  $\square$

**Proposition 4.9.** *If  $S \subseteq R$  is an epimorphic extension, then  $S^G \subseteq R^G$  is an epimorphic extension, if (a)  $R$  is a domain, or (b)  $G$  is finite and  $|G|$  is a unit in  $S$ .*

*Proof.* Let  $H$  and  $H'$  be as in Lemma 4.8. By [16, Lemme 1.0],  $S \subseteq R$  is an epimorphic extension if and only if  $R \cong R \otimes_S R$ . Hence,  $G$  acts on the image of  $R \otimes_S R$  under the canonical isomorphism.

Consider  $\phi : R^G \otimes_{S^G} R^G \rightarrow (R \otimes_S R)^G$  given by  $r \otimes_{S^G} q \mapsto r \otimes_S q$ . Since  $R \cong R \otimes_S R$ , every element of  $R \otimes_S R$  is of the form  $r \otimes_S 1$ . Hence, clearly the  $G$ -action  $\sigma(r \otimes_S 1) = \sigma(r) \otimes_S 1$ , for  $\sigma \in G$ , is well-defined. If  $r \otimes_S 1 \in (R \otimes_S R)^G$ , then for any  $\sigma \in G$  we have  $\sigma(r) \otimes_S 1 = r \otimes_S 1$ , whence  $r = \sigma(r)$ . Thus, there exists  $r \otimes_{S^G} 1 \in R^G \otimes_{S^G} R^G$  such that  $\phi(r \otimes_{S^G} 1) = r \otimes_S 1$ , i.e.  $\phi$  is surjective. Now let  $\sum r_i \otimes_{S^G} q_i$  be a finite sum of simple tensors in  $R^G \otimes_{S^G} R^G$ . Suppose  $\phi(\sum r_i \otimes_{S^G} q_i) = 0$ . Then  $\sum r_i \otimes_S q_i = 0$ . It follows  $\sum (r_i, q_i) \in H$ . Since  $(r_i, q_i) \in F(R^G \times R^G)$ , we have  $\sum (r_i, q_i) \in H \cap F(R^G \times R^G) = H'$ , by Lemma 4.8. Hence,  $\sum r_i \otimes_{S^G} q_i = 0$ . Thus,  $\phi$  is injective.

Lastly we show  $\phi$  is well-defined. Suppose  $r_1 \otimes_{S^G} q_1 = r_2 \otimes_{S^G} q_2$  in  $R^G \otimes_{S^G} R^G$ . Then  $(r_1, q_1) - (r_2, q_2) \in H'$ . Since  $H' \subseteq H$ , we have  $\phi(r_1 \otimes_{S^G} q_1) = r_1 \otimes_S q_1 = r_2 \otimes_S q_2 = \phi(r_2 \otimes_{S^G} q_2)$ . Hence,  $\phi$  is well-defined. Thus,  $R^G \cong R^G \otimes_{S^G} R^G$ ; equivalently,  $S^G \subseteq R^G$  is an epimorphic extension.  $\square$

**Lemma 4.10.** *Let  $I$  be a finitely generated ideal of  $S^G$ . If  $G$  is finite and  $|G|$  is a unit in  $R$ , then  $(IR)^G = IR^G$ .*

*Proof.* Let  $I = (a_1, \dots, a_n)$ , where  $a_i \in S^G$ . Clearly,  $IR^G \subseteq (IR)^G$ . For the reverse containment, let  $x \in (IR)^G$ . Then  $x = r_1 a_1 + \cdots + r_n a_n \in R^G$ , where  $r_i \in R$ . Applying each element of  $G$  to this equation and adding the results we have  $|G|x = \hat{r}_1 a_1 + \cdots + \hat{r}_n a_n$ . Hence,  $x = |G|^{-1}(\hat{r}_1 a_1 + \cdots + \hat{r}_n a_n) \in IR^G$ . Thus,  $(IR)^G = IR^G$ .  $\square$

**Proposition 4.11.** *Assume  $G$  is finite and  $|G|$  is a unit in  $S$ . If  $R$  is  $S$ -flat, then  $R^G$  is  $S^G$ -flat.*

*Proof.* By [15, Proposition 1, p.132], given a ring  $T$ , a  $T$ -module  $M$  is flat if and only if  $I \otimes_T M \cong IM$  canonically, for any finitely generated ideal  $I$  of  $T$ . Let  $I = (a_1, \dots, a_n)$  be an ideal of  $S^G$ . Since  $R$  is  $S$ -flat,  $IS \otimes_S R \cong IR$ , so  $G$  acts on the image of  $IS \otimes_S R$  under the canonical isomorphism. Since every element of  $IS \otimes_S R$  is of the form  $1 \otimes_S x$ , where  $x \in IR$ , clearly the  $G$ -action is given by  $\sigma(1 \otimes_S x) = 1 \otimes_S \sigma(x)$  and is well-defined. By Lemma 4.10,  $(IS \otimes_S R)^G = (IR)^G = IR^G$ . We show  $I \otimes_{S^G} R^G \cong (IS \otimes_S R)^G$ , whence  $I \otimes_{S^G} R^G \cong IR^G$ .

As in Lemma 4.8, let  $H$  and  $H'$  be the subgroups satisfying  $I \otimes_{S^G} R^G = F(I \times R^G)/H'$  and  $(IS \otimes_S R)^G = (F(IS \times R)/H)^G$ . By the same reasoning in Lemma 4.8,  $H' = H \cap F(I \times R^G)$ .

Now consider  $\phi : I \otimes_{S^G} R^G \rightarrow (IS \otimes_S R)^G$  given by  $x \otimes_{S^G} r \mapsto x \otimes_S r$ . Suppose  $\phi(\sum x_i \otimes_{S^G} r_i) = 0$ , where  $\sum x_i \otimes_{S^G} r_i$  is a finite sum of simple tensors in  $I \otimes_{S^G} R^G$ . Then  $\sum x_i \otimes_S r_i = 0$ , so  $\sum (x_i, r_i) \in H$ . Since  $\sum (x_i, r_i) \in F(I \times R^G)$ , we have  $\sum (x_i, r_i) \in H \cap F(I \times R^G) = H'$ . Hence,  $\sum x_i \otimes_{S^G} r_i = 0$ . Thus,  $\phi$  is injective.

Now let  $(s_1 a_1 + \dots + s_n a_n) \otimes_S r \in (IS \otimes_S R)^G$ . Let  $\rho : (IS \otimes_S R)^G \rightarrow (IR)^G$  be the canonical map (isomorphism). Then  $(s_1 a_1 + \dots + s_n a_n) \otimes_S r = \rho^{-1}(r_1 a_1 + \dots + r_n a_n)$ , for some  $r_1 a_1 + \dots + r_n a_n \in (IR)^G = IR^G$ . Note  $a_i \otimes_{S^G} r_i \in I \otimes_{S^G} R^G$  for  $i = 1, \dots, n$ . From this observation it follows

$$\rho(\phi(a_1 \otimes_{S^G} r_1 + \dots + a_n \otimes_{S^G} r_n)) = r_1 a_1 + \dots + r_n a_n.$$

Hence,

$$\begin{aligned} \phi(a_1 \otimes_{S^G} r_1 + \dots + a_n \otimes_{S^G} r_n) &= \rho^{-1}(r_1 a_1 + \dots + r_n a_n) \\ &= (s_1 a_1 + \dots + s_n a_n) \otimes_S r, \end{aligned}$$

since  $\rho$  is an isomorphism. Thus,  $\phi$  is surjective. Hence,  $I \otimes_{S^G} R^G \cong (IS \otimes_S R)^G \cong (IR)^G = IR^G$ . By [15, Proposition 1, p.132],  $R^G$  is  $S^G$ -flat.  $\square$

**Remark 4.12.** We have yet to determine if epimorphic or flat extensions are invariant under locally finite, infinite group action.

Normal pairs are another generalization of integrally closed minimal extensions. As in [4], we say  $(S, R)$  is a *normal pair* if  $T$  is integrally closed in  $R$  for  $T$  any ring between  $S$  and  $R$ .

**Proposition 4.13.** *If  $(S, R)$  is a normal pair, then  $(S^G, R^G)$  is a normal pair, if any of the following conditions hold:*

- (a) *If  $G$  is locally finite,  $R$  is an overring of  $S$ , and  $S$  is both complemented and almost quasilocal.*
- (b) *If  $G$  is finite and  $|G|$  is a unit of  $R$ .*

*Proof.* (a) First note  $R$  is reduced as an overring of a reduced (complemented) ring. It follows  $\text{tq}(S^G) = (\text{tq}(S))^G$ , by [18, Lemma 2.2(2)], whence

$R^G$  is an overring of  $S^G$ . Let  $x \in \text{Reg}(R^G)$ . By [18, Lemma 2.2(1)],  $x \in \text{Reg}(R)$ . It follows  $x \in S$  or  $1/x \in S$ . Hence,  $x \in S^G$  or  $1/x \in S^G$ . By [12, Proposition 3.6],  $(S^G, R^G)$  is normal.

(b) Let  $S^G \subseteq T \subseteq R^G$ , and let  $u \in R^G$  be integral over  $T$ . Then  $u \in R$  is integral over  $TS$ , since  $T \subseteq TS$ . Since  $(S, R)$  is normal,  $u \in TS$ , say  $u = s_1 t_1 + \cdots + s_n t_n$ , where  $s_i \in S$  and  $t_i \in T$ . Applying the averaging technique introduced in Section 2 we have  $|G|u = \widehat{s}_1 t_1 + \cdots + \widehat{s}_n t_n$ . Hence,  $u = |G|^{-1}(\widehat{s}_1 t_1 + \cdots + \widehat{s}_n t_n) \in TS^G = T$ . Thus,  $T$  is integrally closed in  $R^G$ , i.e.  $(S^G, R^G)$  is normal.  $\square$

#### ACKNOWLEDGMENT

The author is immensely grateful to her advisor, Jay Shapiro, for his guidance, suggestions, and help revising the manuscript.

#### REFERENCES

- [1] A. Ayache. Minimal overrings of an integrally closed domain. *Communications in Algebra*, 31:5693–5714, 2003.
- [2] G. M. Bergman. Hereditary commutative rings and centres of hereditary rings. *Proceedings of the London Mathematical Society*, 23:214–236, 1971.
- [3] P.-J. Cahen, D. E. Dobbs, and T. G. Lucas. Characterizing minimal ring extensions. *Rocky Mountain Journal of Mathematics*, 41:1081–1125, 2011.
- [4] E. D. Davis. Overrings of commutative rings, III: Normal pairs. *Transactions of the American Mathematical Society*, 182:175–185, 1973.
- [5] D. E. Dobbs. Every commutative ring has a minimal ring extension. *Communications in Algebra*, 34(10):3875–3881, 2006.
- [6] D. E. Dobbs, B. Mullins, G. Picavet, and M. Picavet-L’Hermitte. On the FIP property for extensions of commutative rings. *Commutative Algebra*, 33:3091–3119, 2005.
- [7] D. E. Dobbs and J. Shapiro. A classification of the minimal ring extensions of an integral domain. *Journal of Algebra*, 305(1):185–193, 2006.
- [8] D. E. Dobbs and J. Shapiro. Descent of divisibility properties of integral domains to fixed rings. *Houston Journal of Mathematics*, 32(2), 2006.
- [9] D. E. Dobbs and J. Shapiro. A classification of the minimal ring extensions of certain commutative rings. *Journal of Algebra*, 308(2):800–821, 2007.
- [10] D. E. Dobbs and J. Shapiro. Descent of minimal overrings of integrally closed domains to fixed rings. *Houston Journal of Mathematics*, 33(1), 2007.
- [11] D. E. Dobbs and J. Shapiro. Transfer of Krull dimension, lying-over, and going-down to the fixed ring. *Communications in Algebra*, 35:1227–1247, 2007.
- [12] D. E. Dobbs and J. Shapiro. Normal pairs with zero-divisors. *Journal of Algebra and its Applications*, 10(2):335–356, 2011.
- [13] D. Ferrand and J.-P. Olivier. Homomorphismes minimaux d’anneaux. *Journal of Algebra*, 16:461–471, 1970.
- [14] I. Kaplansky. *Commutative Rings*. University of Chicago Press, Chicago, revised edition, 1974.
- [15] J. Lambek. *Lectures on Rings and Modules*. Blaisdell, Toronto, 1966.
- [16] D. Lazard. Autour de la platitude. *Bulletin de la Société Mathématique de France*, 97:81–128, 1969.
- [17] M. Manis. Valuations on a commutative ring. *Proceedings of the American Mathematical Society*, 20:193–198, 1969.
- [18] H. Mouanis. A note on fixed rings. *Journal of Algebra and Its Applications*, 4:165–171, 2005.

- [19] G. Picavet and M. Picavet-L'Hermitte. *Multiplicative Ideal Theory in Commutative Algebra*, chapter About Minimal Morphisms, pages 369–386. Springer-Verlag, 2006.
- [20] B. Stenström. *Rings of Quotients*. Springer-Verlag, New York, Heidelberg Berlin, 1975.

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